B.Sc Part I (Hons)

Paper – II

Examples of successive differentiation.

Example (A) If $y = \sin mx + \cos mx$, prove that
$$y_n = m^n \left(1 + (-1)^n \sin 2mx\right)^{\frac{1}{2}}$$

Solution: It is given that $y = \sin mx + \cos mx$.

Therefore from the formula

$$y_n = m^n \sin\left(mx + \frac{n\pi}{2}\right) + m^n \cos\left(mx + \frac{n\pi}{2}\right)$$

$$= m^n \left\{\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right\}.$$

$$= m^n \left[\left\{\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= m^n \left[\sin^2\left(mx + \frac{n\pi}{2}\right) + \cos^2\left(mx + \frac{n\pi}{2}\right) + 2\sin\left(mx + \frac{n\pi}{2}\right)\cos\left(mx + \frac{n\pi}{2}\right)\right]^{1/2}$$

$$= m^n \left[1 + \sin 2\left(mx + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}} = m^n \left[1 + \sin(2mx + n\pi)\right]^{1/2}$$

$$= m^n \left[1 + (\sin 2mx \cdot \cos n\pi + \cos 2mx \cdot \sin n\pi)\right]^{1/2}$$

$$= m^n \left[1 + \cos n\pi \cdot \sin 2mx\right]^{1/2} \quad \text{---} (1)$$

Since $\sin n\pi = 0$

Now $\cos n\pi = -1$ if n is an odd integer.

and $\cos n\pi = +1$ if n is an even integer.

Therefore we can write $\cos n\pi = (-1)^n$

Hence from (1) $y_n = m^n \{1 + (-1)^n \sin 2mx\}^{1/2}$.

Proved

Example (B): If $ax^2 + 2bxy + by^2 = 1$, Prove that
$$\frac{d^2y}{dx^2} = \frac{b^2 - ab}{(bx + by)^3}.$$

Solution: Given $ax^2 + 2bxy + by^2 = 1$.

Differentiating with respect to x, we get

$$a \cdot 2x + 2b\left(x\frac{dy}{dx} + y\right) + b \cdot 2y\frac{dy}{dx} = 0$$

$$\Rightarrow ax + \frac{dy}{dx}(bx+by) + by = 0$$

$$\Rightarrow (ax+by) + \frac{dy}{dx}(bx+by) = 0$$

$$\therefore \frac{dy}{dx} = -\frac{ax+by}{bx+by}$$

Again differentiating w.r.t. respect to x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{(bx+by)(a+by_1) - (ax+by)(b+by_1)}{(bx+by)^2}\right],$$

where $y_1 = \frac{dy}{dx}$.

$$= -\frac{(abx + b^2xy_1 + aby + bbyy_1) - (abx + abxy_1 + b^2y + bbyy_1)}{(bx+by)^2}$$

$$= \frac{(b^2-ab)xy_1 + (ab-b^2)y}{(bx+by)^2}$$

$$= -\frac{(b^2-ab)(xy_1-y)}{(bx+by)^2} = \frac{(b^2-ab)(y-xy_1)}{(bx+by)^2} \quad \text{——(1)}$$

$$\text{But } y - xy_1 = y - x \times \left\{-\frac{ax+by}{bx+by}\right\}$$

$$= y + \frac{ax^2+bxy}{bx+by}$$

$$= \frac{bxy+by^2+ax^2+bxy}{bx+by} = \frac{ax^2+2bxy+by^2}{bx+by}$$

$$= \frac{1}{bx+by}, \text{ Since } ax^2+2bxy+by^2 = 1.$$

therefore, from (1)

$$\frac{d^2y}{dx^2} = \frac{b^2-ab}{(bx+by)^2} \times \frac{1}{bx+by}$$

$$= \frac{b^2-ab}{(bx+by)^3} \quad \underline{\text{Proved}}$$

B.Sc Part I (Hons)

Important Examples (Paper II)

Ex-1. If $y = \sin(m\sin^{-1}x)$, Prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$

Solution:— Here $y = \sin(m\sin^{-1}x)$

$$\therefore y_1 = \cos(m\sin^{-1}x) \times m \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$= m^2(1-\sin^2(m\sin^{-1}x)) = m^2(1-y^2)$$

Now differentiating,

$$(1-x^2)2y_1 \times y_2 + y_1^2(-2x) = m^2(-2yy_1)$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = -m^2y$$

$$\therefore (1-x^2)y_2 - xy_1 + m^2y = 0. \text{ Proved}$$

Ex-(2) If $y = a\cos(\log x) + b\sin(\log x)$ Prove that $x^2y_2 + xy_1 + y = 0$

Solution:— Here $y = a\cos(\log x) + b\sin(\log x)$

$$\therefore y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Again differentiating,

$$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow x^2y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x)$$

$$= -[a\cos(\log x) + b\sin(\log x)] = -y$$

$$\therefore x^2y_2 + xy_1 + y = 0$$

Proved.

Example 3. Find y_n if $y = \frac{1}{x^2+a^2}$.

Solution:-

Here $y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)}$

$= \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$

$\therefore$ from the formula

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n(n)!}{(x-ia)^{n+1}} - \frac{(-1)^n(n)!}{(x+ia)^{n+1}}\right]$$

$$= \frac{(-1)^n(n)!}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

To simplify this, we use De Moivre's Theorem. For this, we put $x = r\cos\theta$ and $a = r\sin\theta$, so that $r^2 = x^2+a^2$ and $\tan\theta = a/x$

Hence $y_n = \frac{(-1)^n n!}{2ia}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$

$$= \frac{(-1)^n n!}{2ia}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}}\left[\frac{1}{(e^{-i\theta})^{n+1}} - \frac{1}{(e^{i\theta})^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}}\left[\frac{1}{e^{-i(n+1)\theta}} - \frac{1}{e^{i(n+1)\theta}}\right] = \frac{(-1)^n n!}{2iar^{n+1}} \cancel{\times 2i\sin(n+1)\theta}$$

$$\left[e^{i(n+1)\theta} - e^{-i(n+1)\theta}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}} \times 2i\sin(n+1)\theta.$$

$$= \frac{(-1)^n n!}{ar^{n+1}} \sin(n+1)\theta.$$

But $r\sin\theta = a$ $\therefore r = \frac{a}{\sin\theta}$

$$\therefore y_n = \frac{(-1)^n n!}{a\left(\frac{a}{\sin\theta}\right)^{n+1}} \cdot \sin(n+1)\theta.$$

i.e. $y_n = \frac{(-1)^n n!}{a^{n+2}} \cdot \sin^{n+1}\theta \sin(n+1)\theta.$

Where $\tan\theta = a/x$ i.e. $\theta = \tan^{-1}\frac{a}{x} = \cot^{-1} x/a.$

Anjani Kumar Singh